Name————————————, ID ——————

Final EXAM , MTH 320, Fall 2016

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QUESTION 1. (i) **(5 points)**. Is $(Q^*, .)$ isomorphic to $(Z, +)$? Explain

No. $(Q^*, .)$ **has a finite group, namely** $\{1, -1\}$**. So** $(Q*, .)$ **is not cyclic (since every subgroup of a cyclic infinite group is cyclic). However,** $(Z, +)$ **is cyclic. Thus** $(Q^*, .)$ **is not isomorphic to** $(Z, +)$**.**

(ii) **(5 points)**. Is $Z_3 \times Z_8$ isomorphic to $Z_6 \times Z_4$? Explain

$Z_3 \times Z_8$ **is isomorphic to** Z_{24} **and hence cyclic. Since** $gcd(6, 4) \neq 1$**,** $Z_6 \times Z_4$ **is not cyclic.**

(iii) **(5 points)** . Let $n = 5^2.7^3.11$, and let $D = \{a \in (Z_n, +) \mid |a| = 77\}$. Find the cardinality of D.

Since Z_n **is cyclic, we know** Z_n **has a unique subgroup of order 77, say** $H =< a >$**. Hence if** $b \in D$**, then** $< a >=< b >$**. Thus** $D = \{c \in H \mid |c| = 77\}$**. We know that** H **has exactly** $\phi(77) = \phi(7 \times 11) = 6 \times 10 = 60$ **elements of order 77. Thus** $|D| = 60$**.**

(iv) **(5 points)**. It is easy to see that A_8 has an elements of order 15. With at most two lines, convince me that A_8 must have at least two distinct subgroups each is of order 15.

Let H **be a subgroup of order 15. Since** A_5 **is simple, there exists** $a \in A_5$ **such that** $a * H \neq H * a$**. Thus** $a * H * a^{-1} \neq H$**. We know** $a * H * a^{-1}$ **is a subgroup of** A_8 **with 15 elements .**

(v) **(5 points)**. Is it possible to have infinitely many non-isomorphic groups such that each has 100 elements? Explain

It is clear that S_{100} **has finitely many subgroups, each is of order 100. By Caley's Theorem a group with 100 elements is isomorphic to a subgroup of** S_{100}**. Thus there are finitely many non-isomorphic groups such that each has 100 elements.**

(vi) **(5 points)**. Give me an example of a group D that has an element w of order 2 and an element f of order 3, but D has no elements of order 6.

S_3 **has no elements of order 6. However** $a = (1\ \ 2)$ **is of order 2 and** $b = (1\ \ 2\ \ 3)$ **is of order 3.**

(vii) **(8 points)**. Let $F : (Z, +) \to (Q^*, .)$ be a nontrivial group homomorphism such that F is not one-to-one. Find $F(1)$, then find $Image(F)$ and $Ker(F)$.

Since F **is not 1-1,** $Ker(f) \neq \{0\}$**. Hence** $Ker(F) = mZ$ **for some** $m \in Z^+$**. Thus** $Z/mZ = Z_m \simeq Image(F) < Q^*$**. Thus** $Image(F)$ **must be finite. However** $(Q^*, .)$ **has a unique finite subgroup** $H = \{1, -1\}$**. Thus** $Image(F) = H \simeq Z_2$**. Hence** $m = 2$ **and** $Ker(F) = 2Z$**. If** $F(1) = 1$**, then** $F(a) = 1$ **for every** $a \in Z$ **and thus** F **is the trivial group homomorphism, a contradiction. Hence** $F(1) = -1$**.**

(viii) **(8 points)**. Let F be a group with 21 elements such that F has a unique subgroup with 3 elements. Prove that F is isomorphic to Z_{21}.

We know F **has a subgroup with 7 elements, say** H**, and it has a subgroup with 3 elements, say** K**. Since** $[H : F] = 3$**, and 3 is the minimum prime divisor of** $|F| = 21$**, we conclude that** $H \lhd F$**. Since** K **is unique, we conclude** $K \lhd F$**. It is clear that** $|HK| = 21$ **and** $H \cap K = \{e\}$**. Hence** $HK = F$ **and F =** $F/(H \cap K) \simeq F/H \times F/K \simeq Z_3 \times Z_7 \simeq Z_{21}$ **is cyclic.**

(ix) **(8 points)**. Let D be a group with 77 elements. Prove that either $|C(D)| = 1$ or D is abelian.

$|C(D) = 1$ **or 7 or 11 or 77. If** $C(D) = 77$**, we are done. If** $C(D) = 7or11$**, then** $D/C(D)$ **is cyclic and hence** D **is abelian.**

(x) **(8 points)**. Let D be a finite group. Assume H is a normal subgroup. Given $|a * H| = n$ (the order of the element $a * H$ is n in G/H) for some $a \in D$. Prove that D has an element of order n.

Let $m = |a|$**. We know** $n \mid m$**. Thus** $m = nk$**. Let** $f = a^k \in D$**. We know** $|f| = |a^k| = \frac{m}{gcd(k,m)} = \frac{m}{k} = n$**.**

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